THE TWENTY-SEVENTH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE twenty-seventh summer meeting and ninth colloquium of the Society were held at the University of Chicago on September 7–11, following the summer meeting of the Mathematical Association of America, which was held on Monday, September 6. The sessions of Tuesday and of Wednesday forenoon were devoted to the reading of papers; the colloquium opened on Wednesday afternoon.

The members of the Society and their guests found excellent arrangements for their entertainment at Hitchcock and Beecher Halls, and at the Quadrangle Club. The Club was very graciously put at the disposal of the members for meals and for use as a social center. On Friday afternoon visits were made under the guidance of Professor Slaught to Ida Noyes Hall and to Harper Memorial Library.

At the joint dinner of the two organizations, one hundred and sixteen persons were present. Professor Slaught, retiring president of the Mathematical Association, presided at the after-dinner speaking, which was participated in by several of those present, to the edification of all the listeners.

The meetings of the Society were attended by more than one hundred and twenty persons, among whom were the following ninety-nine members of the Society:

Professor R. C. Archibald, Professor G. N. Armstrong, Professor I. A. Barnett, Professor Suzan R. Benedict, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor G. A. Bliss, Professor R. L. Borger, Professor J. W. Bradshaw, Professor W. C. Brenke, Professor W. H. Bussey, Professor W. D. Cairns, Professor J. W. Campbell, Professor A. L. Candy, Professor E. W. Chittenden, Professor C. E. Comstock, Professor A. R. Crathorne, Dr. G. H. Cresse, Professor D. R. Curtiss, Professor H. H. Dalaker, Professor S. C. Davisson, Professor E. L. Dodd, Professor L. W. Dowling, Professor Arnold Dresden, Professor Otto Dunkel, Professor M. D. Earle, Professor Arnold Emch, Professor G. C. Evans, Professor H. S. Everett, Professor Peter Field, Professor B. F. Finkel, Professor L. R. Ford, Professor W. B. Ford, Professor M. G. Gaba, Professor W. H. Garrett, Professor

D. C. Gillespie, Professor R. E. Gilman, Dr. T. H. Gronwall, Professor C. F. Gummer, Professor W. L. Hart, Professor M. W. Haskell, Professor Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. A. Hopkins, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. D. C. Kazarinoff, Professor S. D. Killam, Professor H. W. Kuhn, Professor Gillie A. Larew, Professor Flora E. LeStourgeon, Professor A. C. Lunn, Professor E. B. Lytle, Professor R. B. McClenon, Professor J. V. McKelvey, Professor T. E. Mason, Professor Helen A. Merrill, Professor Bessie I. Miller, Professor W. L. Miser, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Professor A. L. Nelson, Professor Anna H. Palmié, Professor Anna J. Pell, Dr. T. A. Pierce, Professor A. D. Pitcher, Professor S. E. Rasor, Professor R. G. D. Richardson, Professor H. L. Rietz, Professor Maria M. Roberts, Professor W. H. Roever, Professor Oscar Schmiedel, Dr. Caroline E. Seely, Professor E. W. Sheldon, Dr. W. G. Simon, Professor E. B. Skinner, Professor H. E. Slaught, Mr. H. L. Smith, Professor E. B. Stouffer, Professor C. E. Stromquist, Professor K. D. Swartzel, Professor E. J. Townsend, Professor A. L. Underhill, Professor E. B. Van Vleck, Professor Oswald Veblen, Dr. J. H. Weaver, Professor W. P. Webber, Mr. F. M. Weida, Professor Mary E. Wells, Professor W. D. A. Westfall, Professor Marion B. White, Professor C. E. Wilder, Professor F. B. Wiley, Professor C. H. Yeaton, Professor J. W. Young, Professor J. W. A. Young.

Vice-President R. G. D. Richardson presided at the meetings of Tuesday and Wednesday forenoons; Professor Haskell presided on Tuesday afternoon.

Upon the recommendation of the Council the Society adopted the following amendments to the By-Laws:

1. By-Law II, section 2, is amended by changing the words "five dollars" to "six dollars," so that the amended section reads "The annual dues shall be six dollars payable on the first of January, etc." Adopted unanimously.

2. By-Law II, section 3, is amended by changing the words "fifty dollars" to "seventy-five dollars," so that the amended section reads "On the payment of seventy-five dollars in one sum, any member of at least four years standing and not in arrears of dues may become a life member and shall thereafter be exempt from all annual dues." Adopted by a vote of 47 to 2.

3. By-Law IX, section 2 is amended by the addition of the sentence: "One member of the Committee of Publication shall be designated by the Council as Editor of the BULLETIN." Adopted unanimously.

The Council announced the election of the following persons to membership in the Society: Dr. R. F. Borden, Brown University; Dr. Tso Chiang, Nan Kai College, Tientsin, China; Professor H. M. Dadourian, Trinity College, Hartford, Conn.; Dr. Jesse Douglas, Columbia University; Mr. Philip Franklin, Princeton University; Dr. C. F. Green, University of Illinois; Captain R. S. Hoar, Ordnance School, Aberdeen, Md.; Dr. Jessie M. Jacobs, University of Texas; Dr. E. L. Post, Princeton University; Professor C. D. Rice, University of Texas; Mr. L. G. Simon, New York City; Professor J. E. Stocker, Lehigh University; Dr. Tsao-Shing Yang, Syracuse University. Twenty-two applications for membership in the Society were received.

The following resolution was adopted unanimously: "The American Mathematical Society expresses to the department of mathematics of the University of Chicago its heartiest appreciation of the arrangements they have made for the twenty-seventh summer meeting of the Society and its gratitude for their hospitality."

Professor Hedrick spoke on behalf of the Committee on increase of membership and of sales of the Society's publications. A considerable number of new subscriptions to the *Transactions* were secured subsequently.

Professor Bliss then offered the following resolution, which was adopted unanimously: "The Society recommends for favorable consideration by the Council applications for membership from advanced students and others interested in mathematics, whether engaged in teaching or not, when properly proposed by members of the Society."

The following papers were read at this meeting:

(1) Professor ARNOLD EMCH: "On the projective generation of cyclides."

(2) Dr. J. H. WEAVER: "A generalization of the strophoid."

(3) Professor C. F. GUMMER: "On the relative distribution of the real roots of two real polynomials."

(4) Professor A. A. BENNETT: "The polyadic expansion of a number."

(5) Dr. J. L. WALSH: "On the location of the roots of the jacobian of two binary forms."

(6) Dr. J. L. WALSH: "On the transformation of convex point sets."

(7) Professor W. B. FORD: "On Kakeya's minimum area problem."

(8) Professor T. H. HILDEBRANDT: "On completely continuous linear transformations."

(9) Professor ANNA J. PELL: "Integral equations in which the kernel is quadratic in the parameter."

(10) Professor OLIVE C. HAZLETT: "Annihilators of modular invariants."

(11) Professors VIRGIL SNYDER and F. R. SHARPE: "Construction of multiple correspondences between two algebraic curves."

(12) Professor DUNHAM JACKSON: "Note on a method of proof in the theory of Fourier's series."

(13) Professor J. W. CAMPBELL: "On the drift of spinning projectiles."

(14) Professor W. L. HART: "Functions of infinitely many variables in Hilbert space."

(15) Professor D. C. GILLESPIE: "A property of continuity."

(16) Professor L. A. HOPKINS: "Periodic orbits of type 2/1."

(17) Professor DUNHAM JACKSON: "Note on the median of a set of numbers."

(18) Professor C. N. MOORE: "An application to Weierstrass's function of the generalized derivative of type (C 1)."

(19) Dr. L. R. FORD: "A method of graduating curves."

(20) Professor E. W. CHITTENDEN: "Note on a generalization of a theorem of Baire."

(21) Professor E. W. CHITTENDEN: "On classes of functions defined in terms of relatively uniform convergence."

(22) Professor E. W. CHITTENDEN: "On the relation between the Hilbert space and the calcul fonctionnel of Fréchet."

(23) Dr. J. L. WALSH: "A generalization of the Fourier cosine series."

(24) Professor DUNHAM JACKSON: "Note on a class of polynomials of approximation."

(25) Professor G. A. MILLER: "Reciprocal subgroups of an abelian group."

(26) Professors E. R. HEDRICK, W. D. A. WESTFALL and LOUIS INGOLD: "Characteristic lines of transformations."

(27) Professor W. L. HART: "Pseudo-differentiation of a summable function with respect to a parameter."

(28) Professor EDWARD KASNER: "Five notes on Einstein's theory of gravitation."

(29) Professor DUNHAM JACKSON: "On the convergence of certain trigonometric approximations."

(30) Professor DUNHAM JACKSON: "Note on the Picard method of successive approximations."

(31) Professor OLIVE C. HAZLETT: "A symbolic notation in the theory of formal modular invariants."

(32) Dr. T. H. GRONWALL: "On the Fourier coefficients of a continuous function."

(33) Dr. T. H. GRONWALL: "A sequence of polynomials connected with the nth roots of unity."

(34) Dr. T. H. GRONWALL: "Upper bounds for the coefficients in conformal mapping."

The papers of Professors Snyder and Sharpe, Dr. Walsh, Professor Miller, Professor Kasner, Professor Chittenden's first paper, Professor Jackson's fifth paper and Dr. Gronwall's first and second papers were read by title. Abstracts of the papers numbered in accordance with the above list of titles are given below.

1. Professor Emch shows how cyclides may be treated from the projective point of view. In the first place it is proved that every cyclide may be generated in an infinite number of ways by two projective pencils of spheres (of which one may degenerate into a pencil of planes). By this method of generation many of the well-known properties of cyclides may easily be derived. It also leads to a generalized cyclide which may be called a quintic cyclide, whose definition is contained in the following theorem:

The tangent planes of a quadric cone and an independent projective pencil of spheres generate a quintic cyclide with a finite circle as a double curve. This quintic degenerates into an ordinary general cyclide when the radical plane of the pencil of spheres coincides with the projectively corresponding tangent plane of the cone.

When α, β, γ are linear quantics representing planes, P and Q quadratic quantics representing spheres, and λ denotes a parameter, the cone of class 2 and the projective pencil of spheres have the form

$$\alpha\lambda^2 + \beta\lambda + \gamma = 0, \quad P - \lambda Q = 0,$$

and the quintic cyclide is

$$\alpha P^2 + \beta PQ + \gamma Q^2 = 0.$$

2. Among the generalizations of the strophoid given by Loria in his treatise Algebraische und transcendente ebene Kurven, the one by W. W. Johnson is as follows: Let there be two fixed points A and B and let two lines l_1 and l_2 make angles φ_1 and φ_2 with the line AB such that

$$m\varphi_1 \pm n\varphi_2 = \alpha \quad (\alpha = \text{constant}).$$

Then if l_1 and l_2 intersect in P, the locus of P is a strophoid.

Since α is a constant there is associated with the strophoid a a circle passing through A and B. Dr. Weaver has developed strophoids having associated with them the three conic sections in the same relation that the circle is associated with the strophoid of Johnson, and has proved a number of theorems on concurrence of lines and collinearity of points connected with the curves.

3. The roots of a real polynomial which lie within a given interval on the real axis are divided into groups by the roots of a second real polynomial. Professor Gummer shows how this distribution may be determined by a rational process. A generating function $\Sigma_r P_r(1+t)^{k-r}(1-t)^r$ is formed such that the coefficients of descending powers of t are the numbers in the successive groups from left to right. Two methods are given for the determination of the P's, corresponding to the theorems of Sturm and Hermite for a single polynomial. Sylvester had given a method for the reduced arrangement in which pairs of roots of either polynomial occurring consecutively were omitted.

4. In this paper Professor Bennett discusses the array consisting of all the p-adic expansions of a given integer, as p ranges through the sequence of primes. For a given p, the p-adic coefficients constitute a row of the array and for a given n, as p varies, the coefficients of p^n constitute a column. For every integer, positive or negative, there is an array, or polyadic expansion, while a polyadic expansion does not in general represent an actual integer. Sections of this array form the natural instruments in the study of numerical modular domains with a composite modulus. Subtraction and division are considered.

5. An annular region is defined as a closed region of the plane bounded by two non-intersecting circles. Dr. Walsh proves that if three annular regions are respectively the envelopes of three points z_1, z_2, z_3, then the envelope of the point z_4 defined by the real constant cross ratio

$$\lambda = (z_1, z_2, z_3, z_4)$$

is also an annular region. This result is applied in giving some geometric results concerning the location of the roots of the jacobian of two binary forms.

If f_1 and f_2 are binary forms of respective degrees p_1 and p_2, then the roots of the jacobian of f_1 and f_2 are the real foci of a certain curve of class $p_1 + p_2 - 1$ which touches all the lines formed by joining the pairs of roots of f_1 and f_2 in all possible ways. The curve has various other interesting properties.

6. The chief purpose of Dr. Walsh's note is to prove that a necessary and sufficient condition that a one-to-one point transformation of the plane or of space transform every convex point set into a convex point set is that it be a collineation.

7. Kakeya's problem in its simplest form is as follows: How should a line segment AB be turned end for end in a plane so as to sweep out a minimum area? Professor Ford shows that there are an infinite number of methods of attaining such a minimum area if it be assumed that, as the segment moves, the area generated nowhere returns into itself; but if it be assumed that it returns into itself exactly once in one or more regions, that is, if duplication be allowed, then there is no minimum area attainable by motions confined to the finite plane.

8. In a paper entitled "Lineare Funktionalgleichungen"* F. Riesz has discussed the linear integral equation in an elegant manner, which on account of its fundamental character seems destined to point the way to new and more general results in allied fields. In the first part of Professor Hildebrandt's paper a simple general basis is provided in which the methods and results of Riesz are valid. The second part of the paper takes up a more detailed study of the inversion of the com-

* *Acta Mathematica*, vol. 41 (1916 , pp. 71–96.

pletely continuous transformation which leads to a generalization of the notion of the solution of adjoint homogenous integral equations, and of pseudo-resolvents.

9. Mrs. Pell considers the questions of the existence of solutions of linear integral equations in which the kernel has the form $\lambda K(x, s) + \lambda^2 L(x, s)$ and K and L satisfy certain conditions, and the expansion of arbitrary functions in terms of the solutions.

10. In any theory of invariants, differential operators which annihilate invariants are useful in the computation of invariants and covariants, and are sometimes important in the development of the general theory. Professor Hazlett's paper determines such annihilators for modular invariants. These operators are in some ways analogous to the well known Aronhold operators in the theory of classical invariants, but are very much more complicated, as might be expected. They are, in fact, of the general type anticipated by Professor Dickson in a paper published in 1907.

11. It is well known (Hurwitz, *Mathematische Annalen*, volume 28) that algebraic correspondences between two algebraic curves exist which require two auxiliary equations for their definition. The condition that two equations shall be required was found by Castelnuovo (*Rendiconti dei Lincei*, 1906), but no illustrations have been given. The example given by Amodeo (*Annali*, series 2, volume 20), cited by Castelnuovo, of the intersection of two ruled surfaces in general position can be defined by one equation. The paper by Professors Snyder and Sharpe discusses this example, and gives various methods of constructing correspondences that require two equations for their definition. The paper will appear in the *Transactions*.

12. Professor Jackson's first paper will appear in a later number of the BULLETIN.

13. Professor Campbell treats the problem of drift for the case in which initially the rotation of the projectile is entirely about the axis. He obtains a formula which exhibits qualitatively the well-known characteristics of the phenomenon. An application to the British Mark VI gives results which are quantitatively consistent with observed values.

14. In considering functions of infinitely many real variables $(x_1, x_2, x_3, \ldots)$, the case which is of most interest as viewed from the standpoint of the theory of integral equations and of more general functional equations is that in which it is supposed that $\Sigma_{i=1}^{\infty} x_i^2$ converges. In the present paper, Professor Hart assumes that the points $\xi = (x_1, x_2, \cdots)$ considered satisfy this condition.

A function $f(\xi)$ is said to be completely continuous at a point $\xi = (a_1, a_2, \cdots)$ if the second of the following equations holds whenever the first does:

$$\lim_{n=\infty} \sum_{j=1}^{\infty} (x_{jn} - a_j)^2 = 0; \tag{1}$$

$$\lim_{n=\infty} f(x_{1n}, x_{2n}, \cdots) = f(a_1, a_2, \cdots). \tag{2}$$

Professor Hart first proves certain general theorems, regarding completely continuous functions $f(\xi)$, including a mean value theorem which gives, as a corollary, an expression for the differential of $f(\xi)$. There is then considered the proof of the existence of a continuous solution of the infinite system of equations

$$f_i(t, x_1, x_2, \cdots) = 0 \qquad (i = 1, 2, \cdots), \tag{3}$$

and the existence of a solution of the system of differential equations

$$\frac{dx_i}{dt} = g_i(t, x_1, x_2, \cdots) \qquad (i = 1, 2, \cdots). \tag{4}$$

In (3) and (4) the f_i and g_i are supposed to be completely continuous in their arguments.

15. A continuous function $f(x)$ has the property that between x_1 and x_2 there is at least one value of x for which $f(x)$ has any prescribed value between $f(x_1)$ and $f(x_2)$. This property, common to continuous functions, is possessed by some discontinuous functions. Professor Gillespie's note is concerned with functions which have this property. Conditions which are sufficient to insure their continuity are developed; the character of the discontinuities that such a function may have is shown; and a function, having this property and having its set of points of continuity and its set of points of discontinuity each everywhere dense, is constructed.

16. By combining the analytic processes of Poincaré with mechanical quadrature, Professor Hopkins has obtained a family of periodic orbits in the restricted problem of three bodies, which has significance in the study of the gap in the asteroids where the period would be one half the period of Jupiter, and in the consideration of Cassini's gap in the rings of Saturn. These orbits constitute an exceptional case in certain work of F. R. Moulton and are related to results of Poincaré, E. W. Brown and G. W. Hill.

17. Professor Jackson's second paper will appear in a later number of the BULLETIN.

18. In this paper Professor Moore shows that under certain conditions Weierstrass's function, $f(x) = \Sigma a^n \cos b^n \pi x$, whose derivative in the ordinary sense fails to exist, has a generalized derivative of type $(C1)$. The definition of such generalized derivatives has been given in two papers previously presented to the Society. (Cf. this BULLETIN, volume 25 (1919), pages 249 and 257.)

19. In the smoothing of irregular curves based on original data a satisfactory combination of smoothness and faithfulness to the data is desired. Professor Ford considers the case of a function $f(x)$, given for equidistant values of the argument $x = 0, 1, \cdots, m$, and determines the graduated function $y(x)$ by minimizing the sum

$$\sum_{x=0}^{m} \{M(x)[y(x) - f(x)]^2 + [\Delta^n y(x)]^2\},$$

where the weight $M(x)$ is positive, and where n is the order of differences it is desired to make as small as possible. This leads to reasonably simple numerical methods, and, if $M(x)$ is constant, the solution has the property, desirable in certain cases of graduation, that its moments, up to the $(n-1)$th, about any point are the same as the corresponding moments of the ungraduated function.

20. Professor Chittenden's first paper appeared in full in the October number of the BULLETIN.

21. Denote by B_0 the class of all continuous functions defined on a limited perfect subset P of n-space; by B_a the class

of all functions which do not belong to any class of order less than α and are limits of sequences of functions of class less than α. The theory of the ordinal series of classes B_α has been developed by Baire, Lebesgue and Vallée-Poussin. In terms of the relatively uniform convergence of E. H. Moore, Professor Chittenden has previously defined the corresponding series of classes R_α and now reports on the results of further investigations.

A set E is of type (α, α) at most if its characteristic function is of class B_α at most. Denote by A_α the class of all functions of class B_α such that for every pair of numbers a, b the set $E = [a < f < b]$ is of type (α, α) at most; and by F_α the set of all functions of class B_α which assume a limited number of distinct values. Then $F_\alpha < A_\alpha$ and we have $F_{\alpha+1} < R_{\alpha+1} \leqq A_{\alpha+1} + B_\alpha$, while if α is of the second kind, then $F_\alpha \leqq R_\alpha \leqq A_\alpha < B_\alpha$. Necessary and sufficient conditions that a function be of class R_α are found; but, except in the case $\alpha \leqq 1$, it has not been determined whether or not A_α contains functions not in R_α. The classes F_α and A_α are investigated, with results of interest in the general theory of the series of classes B_α.

22. It is well known that the points of the Hilbert space Ω of infinitely many dimensions can be placed in one-to-one correspondence with the set Ω_1 of all summable functions on an interval I, if we agree to identify functions of Ω_1 which differ only on a set of points of measure zero. Fréchet has called attention to the fact that the subset φ of Ω which corresponds to the set of all continuous functions on I is not closed under the usual definition of distance for the Hilbert space and suggests the desirability of a definition of distance for that space relative to which the set φ will be closed.

Employing results of Féjer and the theory of convergence in mean, Professor Chittenden defines distance for the Hilbert space in terms of the coordinates of a point and trigonometric functions so that limit in the Hilbert space becomes equivalent to uniform convergence on I, excepting a set of points of measure zero. When the functions considered are continuous this implies the continuity of the limit function and leads to the desired closure of the set φ.

The paper will be published in the *Palermo Rendiconti*.

23. Dr. Walsh proves by use of the theory of infinitely many variables that if we consider the set of functions

$$\frac{1}{\pi}\cos\lambda_0 x,\ \frac{2}{\pi}\cos\lambda_1 x,\ \frac{2}{\pi}\cos\lambda_2 x,\ \frac{2}{\pi}\cos\lambda_3 x,\ \cdots\quad (0 \leqq x \leqq \pi),$$

where

$$\lambda_0^2 + 4(\lambda_1 - 1)^2 + 4(\lambda_2 - 2)^2 + 4(\lambda_3 - 3)^2 + \cdots < \frac{1}{\pi}$$

and where

$$\sum_{n=1}^{\infty} n^2 |\lambda_n - n|$$

converges, then there exists another set of functions $\{v_n(x)\}$ biorthogonal to that set. Moreover, for any function $f(x)$ integrable in the sense of Lebesgue and with an integrable square, the two series

$$\tfrac{1}{2}a_0 + a_1\cos x + a_2\cos 2x + \cdots,\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx dx,$$

$$\frac{b_0}{\pi}\cos\lambda_0 x + \frac{2b_1}{\pi}\cos\lambda_1 x + \frac{2b_2}{\pi}\cos\lambda_2 x + \cdots,$$

$$b_n = \int_0^{\pi} f(x)v_n(x)dx,$$

have essentially the same convergence properties throughout the interval $0 \leqq x \leqq \pi$.

24. In a paper recently presented to the Society (see this BULLETIN, June, 1920, page 391) it was shown that if $f(x)$ is a given continuous function in the interval $a \leqq x \leqq b$, n a given positive integer, and m a given real number greater than 1, there exists one and just one polynomial $\varphi(x)$, of degree n or lower, which reduces to a minimum the value of the integral

$$\int_a^b |f(x) - \varphi(x)|^m dx.$$

The purpose of Professor Jackson's third paper is to establish the truth of the same proposition in the case that $m = 1$. The existence of at least one minimizing polynomial is proved in essentially the same way as before, the details of the argument being somewhat simpler in the present case. The proof of uniqueness is considerably less direct, and appears to involve

elementary considerations of the measure of point sets. A notable difference between the two cases is that for $m > 1$ the remainder corresponding to the polynomial of best approximation must change sign at least $n + 1$ times in the interval, unless it vanishes identically, while for $m = 1$ the assertion is merely that the remainder must change sign at least $n + 1$ times, or else vanish throughout a finite number or an enumerable infinity of intervals contained in (a, b), and having at least a certain specified aggregate length.

25. Any two subgroups of the group G which have the property that the product of their orders is equal to the order of G have been called reciprocal subgroups of G. In the present paper Professor Miller confines his attention to reciprocal subgroups of abelian groups. Among the theorems proved are the following: If two reciprocal subgroups of any abelian group have only identity in common, the number of the conjugates of the one under the group of isomorphisms is equal to the number of the conjugates of the other under this group. Whenever an abelian group of order p^m, p being a prime number, contains subgroups of the same order but of different types, then the number of the subgroups of one and of only one of these types is of the form $1 + kp$. The number of the subgroups of each of the other types is divisible by p. A necessary and sufficient condition that the number of the subgroups of a given type contained in such a group is a power of p is that the number of its independent generators of each order increased by the number of the larger independent generators in the set is equal to the number of the independent generators of G whose orders are not less than this order.

26. In this paper Professors Hedrick, Westfall, and Ingold discuss the properties of a double orthogonal set of lines defined for any transformation that is not conformal by the property that their directions at each point are the directions of maximum and minimum stretching due to the transformation. These lines were originally defined by Tissot by a different property. It is shown that there always exists a transformation which corresponds to any preassigned set of such characteristic lines, and that the rate of stretching along one family may also be preassigned. Other properties are found and special cases are discussed.

27. Consider a function $u(s, t)$, where $a \leqq s \leqq b$ and where t assumes all values in a measurable set E. Let us suppose that, for every value of s, $u(s, t)$ becomes a function of t which, together with its square $u^2(s, t)$, is summable in the Lebesgue sense in E. Professor Hart defines $u_s(s_0, t)$ as the pseudo-derivative function of $u(s, t)$ with respect to s at the point s_0 if it satisfies the condition

$$\lim_{\Delta s=0} \int_E \left[\frac{u(s_0+\Delta s, t) - u(s_0, t)}{\Delta s} - u_s(s_0, t) \right]^2 dt = 0.$$

If $u_s(s_0, t)$ exists, it is unique except for its values at a set of points t of measure zero. If $u_s(s, t)$ exists and if $g(t)$ and $(g(t))^2$ are summable, the function

$$h(s) = \int_E u(s, t)g(t)dt$$

has a derivative dh/ds. It is found that the Fourier constants $[x_1(s), x_2(s), \cdots]$ and $[y_1(s), y_2(s), \cdots]$ of $u(s, t)$ and $u_s(s, t)$, respectively, relative to a complete, unitary orthogonal system of functions $[g_1(t), g_2(t), \cdots]$, satisfy the relations

$$dx_i(s)/ds = y_i(s) \qquad (i = 1, 2, \cdots).$$

Under suitable conditions regarding the summability of the function $u_s(s, t)$ in the rectangle $a \leqq s \leqq b$, $c \leqq t \leqq d$, there is given a generalization of the mean value theorem for the function $u(s, t)$. After a few auxiliary results are established, the solution of a certain type of pseudo-differential functional equations is obtained by means of certain theorems regarding differential equations in infinitely many variables.

28. Professor Kasner's notes all relate to four-dimensional riemannian manifolds $ds^2 = \Sigma g_{ij}dx_i dx_j$ obeying Einstein's ten gravitational equations $R_{ij} = 0$, where the g's are the ten potentials and the equations denote the vanishing of the so-called contracted Riemann-Christoffel tensor (which might appropriately be called the Einstein tensor). The main results follow:

(1) It is known that if the paths of particles *and* light pulses can be regarded (in some coordinate system) as straight lines, then the ds^2 is necessarily equivalent to the euclidean form. The same result is here shown to follow if the hypothesis is limited to particles alone, or to light pulses alone.

(2) Can two Einstein manifolds ever have the same light paths? This is shown to be impossible, at least in the case of approximately euclidean manifolds. Application to the one-body problem (solar gravitation): the Mercury effect could be predicted from the observed light deflection.

(3) Approximately euclidean solutions which can be expressed as the sum of four squares; in particular the quasi-conformal type $A(dx_1^2 + dx_2^2 + dx_3^2) + Bdx_4^2$ (which includes the solar field) in explicit form.

(4) If the coefficients of the four squares are functions of a single variable, the only exact (finite) solutions are certain exponentials or powers.

(5) Discussion of the possible Einstein manifolds which can be regarded as immersed in a flat space of 5 or 6 dimensions. The solar field appears in the latter case.

29. This paper is concerned with the trigonometric sums $T_{mn}(x)$ which furnish the closest approximation to a given continuous periodic function $f(x)$, in the sense of the integral of the mth power of the absolute value of the error, for prescribed values of the exponent m and the order n of the sum. The question at issue is the convergence of $T_{mn}(x)$ to the value $f(x)$, when m is held fast ($m \geqq 1$) and n is allowed to become infinite. Professor Jackson shows that a sufficient condition for uniform convergence is that $\lim_{\delta=0} \omega(\delta)/\sqrt[m]{\delta} = 0$, where $\omega(\delta)$ is the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. For $m = 1$, it is sufficient that $f(x)$ have a continuous derivative. The proof makes use of Bernstein's theorem on the derivative of a trigonometric sum. As the condition obtained is less general, even for large values of m, than the well-known Lipschitz-Dini condition in the case of Fourier's series, $m = 2$, the present results may be regarded as preliminary.

30. In presenting the Picard existence proof for a differential equation $dy/dx = f(x, y)$, it is customary to assume that f is continuous, and satisfies a Lipschitz condition in y, throughout a certain rectangle in the xy-plane, and then to show that the successive approximations converge to a solution of the differential equation throughout a sufficiently restricted interval of values of x. Professor Jackson points out that by extending the definition of f outside the rectangle, so that the continuity conditions are maintained, but otherwise

arbitrarily, the process can be made to converge throughout the entire range of values of x originally considered. A function is obtained which satisfies the *original* differential equation as long as the x and y *of the solution* remain in the rectangle, whatever the behavior of the approximating functions may be. This remark is rather obvious, but it is conspicuously omitted from standard treatments of the subject. It applies equally well to systems of n differential equations in n unknown functions, and to regions that are not rectangular.

31. Professor Hazlett's second paper considers a symbolic notation for the modular invariants of a binary form with respect to the general Galois field $GF[p^n]$ of order p^n. In the classic theory of algebraic invariants, a convenient symbolic notation is obtained by expressing the binary form f of order n symbolically as a perfect nth power, $f = \alpha_x^n = \beta_x^n = \gamma_x^n = \cdots$, where $\alpha_x = \alpha_1 x_1 + \alpha_2 x_2$, $\beta_x = \beta_1 x_1 + \beta_2 x_2$, etc. Such a notation is not, however, practicable in the theory of modular invariants. If we express f as the product of n linear factors which are symbolically distinct, $f = \alpha_x \beta_x \cdots$, we have a symbolic notation by the aid of which we can write down all modular invariants of f, both formal and otherwise. In fact every formal modular invariant can be expressed as a polynomial in a finite number of symbolic expressions which behave like invariants. This theorem is illustrated for the binary cubic modulo 2 and the binary quadratic modulo 3. There is a systematic method by which any formal modular invariant for these two special cases can be expressed in symbolic form, and the writer suspects that such is true in general.

32. Given any real and single valued function $\varphi(x)$ which tends to infinity with x, Dr. Gronwall shows how to construct a function $f(\theta)$ continuous for $0 \leqq \theta \leqq 2\pi$ and such that its Fourier coefficients a_n, b_n (which have the well-known property that $\Sigma(a_n^2 + b_n^2)$ converges) make the series

$$\Sigma(a_n^2 + b_n^2)\varphi\left(\frac{1}{a_n^2 + b_n^2}\right)$$

divergent.

33. Dr. Gronwall considers a polynomial $F(z)$ of degree $n - 1$ which does not exceed unity in absolute value at the nth roots of unity, so that $|F(z)| \leqq 1$ for $z = 1, \epsilon, \epsilon^2, \cdots, \epsilon^{n-1}$

where $\epsilon = e^{2\pi i/n}$. It is shown that on the unit circle $|z| = 1$ we have

$$(1) \qquad |F(z)| < \frac{1}{n}\sum_{\nu=0}^{n-1} \frac{1}{\sin\dfrac{2\nu+1}{2n}\pi}$$

except when $F(z)$ has the form $e^{\alpha i} f(\epsilon^{-k}z)$, where α is real, k an integer and

$$(2) \qquad f(z) = \sum_{\nu=0}^{n-1} \frac{(\epsilon^{-\frac{1}{2}}z)^{\nu}}{n \sin\dfrac{2\nu+1}{2n}\pi},$$

in which case the upper bound of $|F(z)|$ is reached at $z = \epsilon^{\frac{1}{2}+k}$. The polynomial $f(z)$ has all its zeros on the unit circle, one in each of the intervals from ϵ to ϵ^2, ϵ^2 to ϵ^3, $\cdots$, ϵ^{n-1} to 1. The asymptotic value of the upper bound in (1) is

$$\frac{2}{\pi}\left(\log n + C + \log\frac{2}{\pi}\right) + \sigma(1)$$

where C is Euler's constant, and $\sigma(1)$ tends to zero as n increases indefinitely.

34. Dr. Gronwall shows that when $w = z + a_2z^2 + \cdots a_nz^n + \cdots$ maps the circle $|z| < 1$ conformally on a simply connected and nowhere overlapping region in the w-plane, then $|a_n| \leqq n$ for $n = 2, 3, \cdots$. When $|a_n| = n$ for any particular n, then also $|a_n| = n$ for every n, and the function w reduces to the one which gives extreme values to the distortion.

ARNOLD DRESDEN,
Acting Secretary.

THE CHICAGO COLLOQUIUM.

THE ninth colloquium of the American Mathematical Society was held in connection with its twenty-seventh summer meeting at the University of Chicago, September 8–11, 1920. At the annual meeting of 1917, the Council, on the invitation of the Department of Mathematics of the University of Chicago, appointed a committee, consisting of Professors E. H. Moore, E. W. Brown, Max Mason, H. S. White, and the Secretary, to arrange for a summer meeting and

colloquium to be held at Chicago in 1919. At the annual meeting of 1918 the Council authorized the postponement of the Chicago meeting until the summer of 1920. The courses of lectures were announced in the preliminary circular of May, 1920, and printed syllabi were distributed at the meeting. The colloquium opened Wednesday afternoon, September 8, and continued until Saturday noon; three lectures were delivered on Thursday, and Friday, and two on each of the other days. At the close of each lecture opportunity for discussion was given, on points of that or previous lectures.

The following ninety persons were in attendance, a number considerably exceeding that of any previous colloquium.

Mr. E. S. Akeley, Professor R. C. Archibald, Professor G. N. Armstrong, Professor I. A. Barnett, Professor S. R. Benedict, Professor A. A. Bennett, Professor G. D. Birkhoff, Professor G. A. Bliss, Professor R. L. Borger, Professor J. W. Bradshaw, Professor W. C. Brenke, Professor W. H. Bussey, Professor W. D. Cairns, Professor J. W. Campbell, Mr. F. E. Carr, Mr. W. E. Cederberg, Professor E. W. Chittenden, Professor A. R. Crathorne, Dr. G. H. Cresse, Professor D. R. Curtiss, Professor E. L. Dodd, Professor L. W. Dowling, Professor Arnold Dresden, Professor Otto Dunkel, Mr. J. D. Eshleman, Professor G. C. Evans, Professor H. S. Everett, Dr. L. R. Ford, Professor W. B. Ford, Professor M. G. Gaba, Professor D. C. Gillespie, Professor R. E. Gilman, Dr. T. H. Gronwall, Professor C. F. Gummer, Professor W. L. Hart, Professor M. W. Haskell, Professor Olive C. Hazlett, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor L. A. Hopkins, Professor W. A. Hurwitz, Professor Dunham Jackson, Mr. D. C. Kazarinoff, Miss Claribel Kendall, Professor S. D. Killam, Professor H. W. Kuhn, Professor Gillie A. Larew, Professor Flora E. LeStourgeon, Mrs. Mayme I. Logsdon, Professor A. C. Lunn, Mr. C. C. MacDuffee, Professor Helen A. Merrill, Professor Bessie I. Miller, Professor W. L. Miser, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Professor A. L. Nelson, Mr. H. L. Olson, Miss Eleanor Pairman, Professor Anna H. Palmié, Professor Anna J. Pell, Dr. T. A. Pierce, Professor A. D. Pitcher, Professor S. E. Rasor, Professor R. G. D. Richardson, Professor H. L. Rietz, Professor W. H. Roever, Miss I. M. Schottenfels, Dr. Caroline Seely, Professor E. W. Sheldon, Dr. W. G. Simon, Professor E. B. Skinner,

Professor H. E. Slaught, Professor H. L. Smith, Professor C. E. Stromquist, Professor K. D. Swartzel, Dr. Bird M. Turner, Professor A. L. Underhill, Professor E. B. Van Vleck, Mr. J. H. Van Vleck, Professor Oswald Veblen, Professor W. P. Webber, Mr. F. M. Weida, Professor Mary E. Wells, Professor C. E. Wilder, Professor F. B. Wiley, Professor C. H. Yeaton, Professor J. W. A. Young.

Two courses of five lectures each were given. as follows:

I. Professor G. D. BIRKHOFF: "Dynamical systems."

II. Professor F. R. MOULTON: "Topics from the theory of functions of infinitely many variables."

Abstracts of the lectures follow below. The lectures will be published later in full as Volume VI of the Colloquium Series.

I.

LECTURE I. PHYSICAL, FORMAL, AND COMPUTATIONAL ASPECTS OF DYNAMICAL SYSTEMS.

1. The conservation of energy; rates of doing work of external forces:

$$Q_i = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + R_i \qquad (i = 1, 2, \cdots, n),$$

$$\sum_{i=1}^{n} \dot{q}_i R_i \equiv 0.$$

2. Lagrangian systems; $R_1 = R_2 = \cdots = R_n = 0$. Internal and external characterization.

3. Variational form of equations under no external forces: $\delta \int L dt = 0$. Change of variables.

4. Equations of variation.

5. Integrals linear or quadratic in the velocities.

6. The Hamiltonian equations: $\frac{dp_i}{dt} = \frac{\partial H}{\partial q_i}$; $\frac{dq_i}{dt} = -\frac{\partial H}{\partial p_i}$.

Their fundamental properties. Formal series.

7. Methods of computation and their validity.

8. Relativistic dynamics.

9. Dissipative systems.

LECTURE II. TYPES OF MOTIONS SUCH AS PERIODIC AND RECURRENT MOTIONS, AND MOTIONS ASYMPTOTIC TO THEM.

1. Existence of periodic motions; minimum and minimax methods; method of analytic continuation; other methods.
2. Hyperbolic periodic motions and their asymptotic motions.
3. Stable elliptic periodic motions.
4. Unstable elliptic periodic motions and their asymptotic motions.
5. Recurrent motions and their asymptotic motions.
6. Other types of motions.
7. The extension in General Analysis.

LECTURE III. INTERRELATION OF TYPES OF MOTION WITH PARTICULAR REFERENCE TO INTEGRABILITY AND STABILITY.

1. Transitivity and intransitivity.
2. Distribution of periodic motions.
3. Distribution of recurrent and other special motions.
4. Criteria for various types of integrability.
5. Non-integrability of general case.
6. Criteria for various types of stability.

LECTURE IV. THE PROBLEM OF THREE BODIES AND ITS EXTENSION.

1. The equations of motion and the classical integrals.
2. Regularization at double collision.
3. Impossibility of triple collision if area constants are not all zero.
4. Proof that for given area constants not all zero, and small initial mutual distances, the sum of the mutual distances becomes infinite.
5. Deduction of Sundman's theorem on mutual distances as corollary.
6. Extension to more general law of force.
7. Extension to case of n bodies.

LECTURE V. THE SIGNIFICANCE OF DYNAMICAL SYSTEMS FOR GENERAL SCIENTIFIC THOUGHT.

1. The dynamical model in physics.
2. Modern cosmogony and dynamics.
3. Dynamics and biological thought.
4. Dynamics and philosophical speculation.

II.

LECTURE I. INFINITE SYSTEMS OF LINEAR EQUATIONS.

1. Completely reduced systems. Historical examples.
2. The formal method of reduced systems. Historical examples.
3. Normal infinite determinants. The Hill-Poincaré form; the von Koch form.
4. Infinite systems of linear equations having normal determinants and bounded right members.
5. Absolutely convergent infinite determinants.
6. Infinite systems of linear equations having absolutely converging determinants and bounded right members.
7. Infinite systems of linear equations having absolutely converging determinants and coefficients analytic functions of a parameter.
8. Irregular solutions of infinite systems of linear equations. Examples.
9. The general theory of Schmidt. Solutions for which

$$\sum_{i=1}^{\infty} |x_i|^p \qquad (p > 1)$$

converges, including the limiting cases $p = 1$, $p = \infty$.
10. The method of successive approximations.

LECTURE II. ON PROPERTIES OF FUNCTIONS OF INFINITELY MANY VARIABLES.

1. Hilbert space (H-space) and parallelepipedon space (P-space). The mutual independence of H-space and P-space.
2. Definitions of limit points. Existence of a limit point in an infinite set of points.
3. Types of continuity and relations among them.
4. Convergent functions and uniformly convergent functions.

5. Independence of continuity and convergence.

6. Representation of convergent functions by series.

7. A continuous function of infinitely many variables in a closed P-space has a maximum and a minimum and all intermediate values.

8. Definition of analytic functions of infinitely many variables.

9. Definition of normal functions of infinitely many variables.

10. Finite operations on functions of infinitely many variables.

11. Limiting processes on functions of infinitely many variables.

12. The mean-value theorem for completely continuous functions having first derivatives.

13. Taylor's theorem for functions of infinitely many variables.

Lecture III. Infinite Systems of Implicit Function Equations.

1. Analytic solutions of reduced normal equations.

2. Analytic continuation of the solutions.

3. Solutions of normal equations having normal determinants of the coefficients of the linear terms of the dependent variables.

4. Properties of the solutions of normal equations.

5. Solution of reduced normal equations by the method of successive approximations.

6. Extension of the solution to a boundary of the region of definition of the equations.

7. Solutions of infinite systems of equations having continuous first derivatives.

8. Properties of the solutions.

9. Extension of the solution to a boundary of the region of definition of the equations.

Lecture IV. Infinite Systems of Differential Equations.

1. Analytic solutions of normal equations.

2. Solution of normal equations by the method of successive approximations.

3. Solution of equations satisfying the Lipschitz condition by the method of successive approximations.

4. Properties of the solutions.

5. Extension of the solution to a boundary of the region for which the equations are defined.

6. Solution by the Cauchy-Lipschitz method.

7. Solutions of infinite systems of linear differential equations having constant coefficients.

8. Solutions of infinite systems of linear differential equations having periodic coefficients.

LECTURE V. APPLICATIONS OF FUNCTIONS OF INFINITELY MANY VARIABLES.

1. Hill's problem of the motion of the lunar perigee.

2. Solutions of linear differential equations in the vicinity of singular points.

3. The determination of the moon's variational orbit.

4. Determination of periodic solutions of certain finite systems of differential equations.

5. The dynamics of a certain type of infinite universe.

At the close of the colloquium, Professor E. B. Van Vleck expressed the appreciation of those present for the excellence of the lectures, and tendered the thanks of the American Mathematical Society to the University of Chicago for the generous provision it had made for the colloquium, and for the welfare of the participants. An appropriate reply was made by Professor E. H. Moore.

W. A. HURWITZ.

NOTE ON VELOCITY SYSTEMS IN CURVED SPACE OF N DIMENSIONS.

BY PROFESSOR JOSEPH LIPKA.

(Read before the American Mathematical Society April 24, 1920.)

§ 1. *Introduction.*

IN a previous paper,* the author gave a complete geometric characterization of the families of curves (termed natural

* "Natural families of curves in a general curved space of n dimensions," *Trans. Amer. Math. Society*, vol. 13 (1912), pp. 77–95. We shall hereafter refer to this paper as "Natural families."

families) defined as the extremals connected with variation problems of the form

$$\int F ds = \text{minimum}, \tag{1}$$

where F is any point function and ds is the element of arc in the space considered. Such a system consists of $\infty^{2(n-1)}$ curves, one through each point in each direction. Among the dynamical systems whose determination leads to an integral of this form we may mention: (1) the trajectories in a conservative field of force for a given constant of energy h, where $F = \sqrt{W + h}$, W being the work function (negative potential); (2) the brachistochrones under conservative forces, $F = 1/\sqrt{W + h}$; (3) the forms of equilibrium of a homogeneous, flexible, inextensible string acted on by conservative forces (general catenaries), $F = W + h$; the paths of light in an isotropic medium, $F = \nu$, the variable index of refraction.

The complete characteristic geometrical properties of a natural family in any curved space, V_n, are:*

(A_1) The locus of the centers of geodesic curvature of the ∞^{n-1} curves which pass through any point of V_n is a euclidean space of $n - 1$ dimensions (S_{n-1}).

(A_2) The osculating geodesic surfaces (V_2's) at any point of V_n form a bundle of surfaces, i.e., all contain a fixed direction (and hence the geodesic in that direction) which is normal to the S_{n-1} of property A_1.

(B) The n directions at any point of V_n, in which, as a consequence of property A (i.e., A_1 and A_2), the osculating geodesic circles (circles of constant geodesic curvature) hyperosculate the curves of the given family, are mutually orthogonal.

Now, property A alone completely characterizes a much wider class of curves, designated as a velocity system.† We have pointed out several dynamical problems which lead to a system of curves characterized by properties A and B. It is the purpose of this note to point out a dynamical problem which leads to the more general system of curves characterized by property A alone.‡

* "Natural families," p. 78.

† This was designated as a system of type (G) in "Natural families," pp. 85–86.

‡ See the discussion of the problem for a euclidean space of 3 dimensions by E. Kasner, Princeton Colloquium Lectures, p. 42.

§ 2. *Differential Equations of Velocity Systems.*

Let us express our problem analytically. If the element of arc length in a general curved space V_n is given by*

$$ds^2 = \sum_{ik} a_{ik} dx_i dx_k \tag{2}$$

and we use s as the parameter along our curves, the differential equations of any natural system (characterized by properties A and B) are†

$$x_i'' + \sum_{\lambda\mu} \left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\} x_\lambda' x_\mu' = \sum_l \frac{\partial(\log F)}{\partial x_l} (A_{il} - x_i' x_l') \qquad (i = 1, 2, \cdots, n), \tag{3}$$

where A_{il} denotes the minor of a_{il} in the determinant $a = |a_{\lambda\mu}|$ divided by a itself, and we have used the Christoffel symbols

$$\left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\} = \sum_l A_{il} \left[\begin{matrix} \lambda\mu \\ l \end{matrix} \right]; \qquad \left[\begin{matrix} \lambda\mu \\ l \end{matrix} \right] = \sum_i a_{il} \left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\}; \tag{4}$$

$$\left[\begin{matrix} \lambda\mu \\ l \end{matrix} \right] = \frac{1}{2} \left(\frac{\partial a_{\lambda l}}{\partial x_\mu} + \frac{\partial a_{\mu l}}{\partial x_\lambda} - \frac{\partial a_{\lambda\mu}}{\partial x_l} \right)$$

On the other hand, the differential equations of a velocity system (characterized by property A) are‡

$$x_i'' + \sum_{\lambda\mu} \left\{ \begin{matrix} \lambda\mu \\ i \end{matrix} \right\} x_\lambda' x_\mu' = \sum_l \phi_l (A_{il} - x_i' x_l') \qquad (i = 1, 2, \cdots, n), \tag{5}$$

where the ϕ's are arbitrary point functions. This system will reduce to a natural system if

$$\phi_l = \frac{\partial(\log F)}{\partial x_l} \qquad (l = 1, 2, \cdots, n), \tag{6}$$

i.e., the ϕ's are the partial derivatives with respect to x of a single function. This is the analytic equivalent of property B.

* We write only Σ_{ik} and understand that the summation is to be carried out from 1 to n for each of the indicated subscripts.

† "Natural families," p. 80. Throughout this paper, primes refer to total derivatives with respect to arc length s, while dots refer to total derivatives with respect to time t.

‡ "Natural families," p. 85.

§ 3. *Dynamical Interpretation of Velocity System.*

Let us consider the motion of a particle in a curved n-space V_n under any positional forces. We start with the Lagrangian equations of motion,*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_l}\right) - \frac{\partial T}{\partial x_l} = X_l \quad (l = 1, 2, \cdots, n), \tag{7}$$

where T is the kinetic energy, given by

$$T = \tfrac{1}{2}\sum_{ik} a_{ik}\dot{x}_i\dot{x}_k, \tag{8}$$

and the X's are the components of force given as functions of the coordinates $x_1, x_2, \cdots, x_n$. Equations (7) may be expanded as

$$\begin{aligned} X_l &= \frac{d}{dt}\left(\sum_k a_{lk}\dot{x}_k\right) - \frac{1}{2}\sum_{ik}\frac{\partial a_{ik}}{\partial x_l}\dot{x}_i\dot{x}_k, \\ &= \sum_k a_{lk}\ddot{x}_k + \sum_{ik}\frac{\partial a_{lk}}{\partial x_i}\dot{x}_i\dot{x}_k - \frac{1}{2}\sum_{ik}\frac{\partial a_{ik}}{\partial x_l}\dot{x}_i\dot{x}_l, \\ &= \sum_k a_{lk}\ddot{x}_k + \sum_{ik}\left[\begin{matrix} ik \\ l \end{matrix}\right]\dot{x}_i\dot{x}_k. \end{aligned}$$

Multiplying by A_{ml}, summing with respect to l, and employing

$$\sum_i a_{ik}A_{il} = \left\{\begin{matrix} 0, \text{ if } k \neq l \\ 1, \text{ if } k = l \end{matrix}\right\}, \tag{9}$$

we get

$$\ddot{x}_m + \sum_{ik}\left\{\begin{matrix} ik \\ m \end{matrix}\right\}\dot{x}_i\dot{x}_k = \sum_l A_{ml}X_l \quad (m = 1, 2, \cdots, n). \tag{10}$$

These equations give us the components of acceleration along a curve as functions of the coordinates and the components of velocity.

Since the velocity along the curve is given by

$$\dot{s}^2 = \sum_{ik} a_{ik}\dot{x}_i\dot{x}_k, \tag{11}$$

we may, by differentiation, get the acceleration along the path; thus

* See E. T. Whittaker, Analytical Dynamics, p. 39.

$$
\begin{aligned}
\dot{s}\ddot{s} &= \sum_{ik} a_{ik}\dot{x}_i\ddot{x}_k + \frac{1}{2}\sum_{ikr}\frac{\partial a_{ik}}{\partial x_r}\dot{x}_i\dot{x}_k\dot{x}_r \\
&= \sum_{ik} a_{ik}\dot{x}_i\left(\sum_l A_{kl}X_l - \sum_{\alpha\beta}\left\{\begin{matrix}\alpha\beta\\ k\end{matrix}\right\}\dot{x}_\alpha\dot{x}_\beta\right) + \frac{1}{2}\sum_{ikr}\frac{\partial a_{ik}}{\partial x_r}\dot{x}_i\dot{x}_k\dot{x}_r \\
&= \sum_l \dot{x}_l X_l - \sum_{i\alpha\beta}\left[\begin{matrix}\alpha\beta\\ i\end{matrix}\right]\dot{x}_i\dot{x}_\alpha\dot{x}_\beta + \frac{1}{2}\sum_{i\alpha\beta}\frac{\partial a_{i\alpha}}{\partial x_\beta}\dot{x}_i\dot{x}_\alpha\dot{x}_\beta \\
&= \sum_l \dot{x}_l X_l,
\end{aligned}
\tag{12}
$$

the reductions being accomplished by using (4) and (9).

As a first step in getting the differential equations of the trajectories, we have

$$
x_m' = \frac{\dot{x}_m}{\dot{s}}; \qquad x_m'' = \frac{\dot{s}\ddot{x}_m - \dot{x}_m\ddot{s}}{\dot{s}^3};
$$

and using (10) and (12), we get

$$
x_m'' = \frac{1}{\dot{s}^2}\sum_l A_{ml}X_l - \sum_{ik}\left\{\begin{matrix} ik\\ m\end{matrix}\right\}x_i'x_k' - \frac{1}{\dot{s}^2}x_m'\sum_l x_l'X_l,
$$

or

$$
x_m'' + \sum_{ik}\left\{\begin{matrix} ik\\ m\end{matrix}\right\}x_i'x_k' = \frac{1}{\dot{s}^2}\sum_l X_l(A_{ml} - x_m'x_l') \qquad (m = 1, 2, \cdots, n). \tag{13}
$$

To get the differential equations of the trajectories we should have to eliminate the speed $\dot{s}$; this would lead to a set of equations of the third order representing ∞^{2n-1} curves. But, for our purpose, we need not go any further. Equations (13) hold for any trajectory, and along this the speed $\dot{s}$ varies from point to point. Now if in (13) we replace $1/\dot{s}^2$ by a constant c, we get

$$
x_m'' + \sum_{ik}\left\{\begin{matrix} ik\\ m\end{matrix}\right\}x_i'x_k' = c\sum_l X_l(A_{ml} - x_m'x_l') \qquad (m = 1, 2, \cdots, n), \tag{14}
$$

a set of differential equations of the second order representing a system of $\infty^{2(n-1)}$ curves (called a velocity system) one through each point in each direction. This system may therefore be defined dynamically as follows:

A curve is a velocity curve corresponding to the speed $\dot{s}_0$ if a particle starting from a point of such a curve and in the direction of the curve and with that speed describes a trajectory osculating the curve.

Now, we note that equations (14) are, with a change in subscripts, exactly equations (5), where

$$\phi_l = cX_l = \frac{1}{\dot{s}^2} X_l \qquad (l = 1, 2, \cdots, n). \tag{15}$$

We have thus formulated a dynamical problem which leads to the system of curves (called a velocity system) characterized geometrically by property *A*. For each constant value assigned to the speed $\dot{s}$, we get a velocity system, and the totality of ∞^1 systems obtained by varying $\dot{s}$ constitute a complete velocity system of ∞^{2n-1} curves in V_n.

§ 4. *Velocity Systems and Natural Systems.*

Velocity systems are not in general systems of trajectories, brachistochrones, or catenaries, but if the field of force is conservative, then

$$X_l = \frac{\partial W_1}{\partial x_l}, \qquad (l = 1, 2, \cdots, n), \tag{16}$$

where W_1 is the work function defining the field, and, as pointed out in § 2, the velocity system corresponding to a speed $\dot{s}_0$ becomes a natural system defined by equation (1) or by the point function F, where by (6)

$$\phi_l = \frac{\partial(\log F)}{\partial x_l},$$

and by (15)

$$c\frac{\partial W_1}{\partial x_l} = \frac{\partial(\log F)}{\partial x_l};$$

hence

$$F = e^{W_1/\dot{s}^2} \tag{17}$$

On the other hand, in a conservative field of force with work function W_2 and given constant of energy h, the natural system defined by equation (1) or by the point function F is a system of

$$\text{(18)}\qquad \begin{aligned} &\text{trajectories,} && \text{if } F = \sqrt{W_2 + h},\\ &\text{brachistochrones,} && \text{if } F = 1/\sqrt{W_2 + h},\\ &\text{catenaries,} && \text{if } F = W_2 + h. \end{aligned}$$

By comparison of (17) and (18), we may now state:

A velocity system for the speed s_0 in a conservative field with work function W_1 is a system of (1) *trajectories,* (2) *brachistochrones,* (3) *catenaries for the constant of energy h in a conservative field with work function W_2, where*

$$(1)\ W_2 = e^{2W_1/s_0^2} - h, \qquad (2)\ W_2 = e^{-(2W_1/s_0^2)} - h,$$

$$(3)\ W_2 = e^{W_1/s_0^2} - h.$$

Since W_1 = constant gives W_2 = constant, the two fields have the same equipotential hypersurfaces and the same lines of force.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
February, 1920.

AUGUSTUS DE MORGAN ON DIVERGENT SERIES.

BY PROFESSOR FLORIAN CAJORI.

(Read before the San Francisco Section of the American Mathematical Society April 10, 1920.)

SEVERAL English mathematicians writing in the second quarter of the nineteenth century disapproved of the banishment of divergent series which had been brought about by the followers of A. L. Cauchy and N. H. Abel. These protests were unheeded, doubtless because they were not accompanied by indications disclosing how divergent series could be used with safety. There was one exception, however: Augustus De Morgan reached results which, had they been followed up promptly, might have re-introduced divergent series thirty years earlier than was actually the case. De Morgan's researches have been overlooked in historical statements, except by H. Burkhardt,* who, however, missed the parts of De Morgan which foreshadow a new theory.

* H. Burkhardt "Ueber den Gebrauch divergenter Reihen in der Zeit von 1750–1860," *Math. Annalen*, vol. 70 (1911), pp. 169–206. This article contains much minute information regarding many writers.

Not only did De Morgan issue a vigorous protest against the abandonment of divergent series, but he laid the foundation to three parts of the subject as it stands to-day: (1) A principle of divergent series recently enunciated more clearly by G. H. Hardy; (2) a summation formula more general than that of Cauchy; (3) the use of asymptotic series.

Omitting many clever observations made by De Morgan, we confine ourselves to these three parts. There is considerable incompleteness in De Morgan's development of them, especially of the first part.

1. *A Principle of Divergent Series.*—De Morgan's point of view is partly contained in the following assumption*: "If then V be expanded into the series $P_0 + P_1 + P_2 + \cdots$ and if the sum of n terms, $P_0 + P_1 + \cdots + P_{n-1}$ be called Q_n; we obviously have

$$\int_0^a V\,dv = \int_0^a P_0 dv + \int_0^a P_1 dv + \cdots + \int_0^a (V - Q_n)dv,$$

where n is made infinite after integration. When the series $P_0 + P_1 + \cdots$ is convergent, then, even granting that $\int (V - Q_\infty)dv$ may have circumstances peculiar to $n = \infty$, it is of no consequence, since considerations of form are rendered useless by evanescence of value: the elements of $\int (V - Q_n)dv$ must, by the hypothesis of convergency, diminish without limit as compared with the corresponding elements of $\int P_0 dv$, $\int P_1 dv$, etc. Even if integration converted the convergent series into a divergent one, this would still be the case."

In this rather difficult passage, De Morgan considers

$$\int_0^a \lim_{n\to\infty} (V - Q_n)dv \text{ and } \lim_{n\to\infty} \int_0^a (V - Q_n)dv.$$

If $P_0 + P_1 + \cdots$ is convergent, then, even if this infinite series becomes divergent on integration, he finds $\int_0^a \lim_{n\to\infty} (V - Q_n)dv = 0$,

* A. De Morgan "On divergent series, and various points in analysis connected with them" in *Cambridge Philosophical Society, Transactions*, vol. 8, Part II, pp. 182–203; see page 189. The paper bears the date of Jan. 15, 1843; it was read March 4, 1844. De Morgan treated divergent series also in vol. 11, pp. 190–202, of the above *Transactions* and in his Differential and Integral Calculus, London, 1842, chapters 19 and 20, but he did not reach noteworthy results other than those given in his article of 1844.

and the elements of $\int_0^\infty (V - Q_n)dv$ must "diminish without limit" as "n is made infinite after integration." It would seem that $\int_0^\infty \lim_{n\to\infty} (V - Q_n)dv$ is used here for the purpose of interpreting $\lim_{n\to\infty} \int_0^\infty (V - Q_n)dv$. No example is given of the conversion of a convergent into a divergent series by integration, but he takes

$$(1) \quad \frac{1 - c\cos av}{1 - 2c\cos av + c^2} = 1 + c\cos av + \cdots + c^n \cos nav + R_{n+1},$$

then "neglects" R_{n+1} and lets $n = \infty$. Thus he obtains an infinite series which is divergent for $c > 1$. Multiplying both sides by $e^{-v^2}dv$ and integrating from $v = 0$ to $v = \infty$, he gets a convergent series. He shows this result to be arithmetically wrong.

Reversing the order of procedure and retaining R_{n+1} in (1), he integrates first and then lets $n = \infty$; he obtains a convergent result which is arithmetically correct. We have here a distant approach to the principle advanced by G. H. Hardy* nearly sixty years later, to the effect that, in such cases, "we may use the otherwise meaningless expression" first obtained "as a formal equivalent for the determinate expression" obtained last.

2. *A Summation Formula More General than that of Cauchy.*— That De Morgan in 1843 held views which were in advance of his time is evident from the following quotation:† "In every convergent series, the limit of the sum of all its terms *is* the mean value obtained from all the summations: the mean of n partial summations $A_1, (A_1 + A_2), \cdots, (A_1 + A_2 + \cdots + A_n)$ is

$$A_1 + \frac{n-1}{n} A_2 + \frac{n-2}{n} A_3 + \cdots + \frac{1}{n} A_n,$$

which, as n is increased without limit, has $A_1 + A_2 + \cdots$ *ad inf.* for its limit. Hence, by Poisson's principle, by which

* *Cambridge Philosophical Society, Transactions*, vol. 19, 1900–1904, p. 297. See also T. J. I. Bromwich, Infinite Series, London, 1908, p. 267, § 99.

† *Cambridge Philosophical Society, Transactions*, vol. 8, Part II, 1844, p. 192.

I mean the assumption of the right to apply the maxim, 'that which is quantitatively true up to the limit, is true in the same sense at the limit, when the limit presents an *incalculable form*'—we may assert most positively, that $1 - 1 + 1 - \cdots$ must be $\frac{1}{2}$ whenever it is the limiting form of convergency: not on the metaphysical doctrine (probably suggested by the known result) of Leibnitz, namely, that we can see no reason to prefer 0 to 1, or 1 to 0, and must therefore take a mean; but because n partial summations give the mean $1/n \times n/2$ or $1/n \times (n + 1)/2$ according as n is even or odd, and the limit of both is $\frac{1}{2}$."

De Morgan's definition of "sum" is substantially the same as a definition given by G. Frobenius* in 1880 and the first of several definitions given by E. Cesàro† in 1890. De Morgan pointed out that, when applied to convergent series, his formula yields the same results as does the restricted formula ordinarily used for convergent series; that is, he recognized the need of "consistency" of definition.

3. *The Use of Asymptotic Series.*—As De Morgan's paper is dated January 15, 1843, he had not seen Cauchy's article‡ printed later in the same year in which Stirling's divergent series for $\log \Gamma(x)$ is used for computing $\log \Gamma(x)$ for large positive values of x. De Morgan appreciates the importance of asymptotic series, but does not compute the asymptotic value of any one series as was done by Cauchy. Instead, De Morgan is groping after a general theory. He says:§ "When an alternating series is convergent, and a certain number of its terms are taken as an approximation, the first term neglected is a superior limit of the error of approximation. This very useful property was *observed* to belong to large classes of alternating series, when finitely or even infinitely divergent: I do not remember that any one has *denied* that it is universally true, while many have implicitly asserted it. When the series is convergent for a certain number of terms, particularly if the terms become very small before they begin to increase again, it obviously makes the divergent alternating

* G. Frobenius, "Ueber die Leibnitzsche Reihe," *Jour. für Math.*, vol. 89 (1880), p. 262.

† E. Cesàro, "Sur la multiplication des séries," *Bull. des Sciences Math* (2), vol. 14 (1890), p. 119.

‡ Cauchy, *Comptes Rendus*, vol. 17 (28 août, 1843), p. 370; Œuvres, Série I, vol. 8, p. 18.

§ *Cambridge Philosophical Society, Transactions*, vol. 8, Part II, p. 193.

series practically as useful as the converging series, perhaps even more so, for it is very frequent that the greater the ultimate divergence, the greater also is the primitive tendency towards convergence."

The theorem that "the first term neglected is a superior limit of the error of approximation," though, as De Morgan says, not universally true, is true, he says, of large classes of alternating series, including the series $\phi(x) - \phi(x+1) + \phi(x+2) - \cdots$ "for all cases in which $\phi(x)$ can be the expressed by $\int_\alpha^\beta e^{vx} X_v dv$, X_v being always positive between limits."

In the development of the modern theories of divergent series, Augustus De Morgan deserves to be ranked as a pioneer.

On December 23, 1857, Sir William R. Hamilton* wrote to De Morgan: "About diverging series, you know a great deal more than I do. In fact you are aware that I early conceived a sort of prejudice against them, in consequence of some of Poisson's remarks. Counter-remarks of yours had staggered me, but had not been carefully weighed. At last (and, I regret to say it, without having yet found the Papers by you and Stokes on *such* series, for Stokes, or Adams for him, sent me about a month ago a duplicate of his memoir on the numerical calculation of the values of certain definite integrals, having a great affinity to my last Paper) I am become a convert to those Divergents; so far at least as to be satisfied that in an extensive class of cases, and with suitable limitations, they may be safely and advantageously used."

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RUSSELL'S INTRODUCTION TO MATHEMATICAL PHILOSOPHY.

Introduction to Mathematical Philosophy. By BERTRAND RUSSELL. (The Library of Philosophy.) London, Allen and Unwin, and New York, The Macmillan Company, 1919. 8vo. viii + 208 pp. $3.00.

THIS book, called an introduction to mathematical philosophy, is an excellent introduction to that field and, more

* R. P. Graves, Life of Sir William Rowan Hamilton, vol. 3, 1899, p. 538.

particularly, to mathematical logic. In the preface the author brings out the fact that mathematical logic is relevant to philosophy and "for this reason, as well as on account of the intrinsic importance of the subject, some purpose may be served by a succinct account of the main results of mathematical logic in a form requiring neither a knowledge of mathematics nor an aptitude for mathematical symbolism."

The first chapter is concerned with the logical basis of the series of natural numbers. The system of postulates of Peano is discussed in some detail. The postulates used in "arithmetization" are indefinite and there is an increase in definiteness produced by "logicizing" mathematics. We cannot, by Peano's method, explain what we mean by the undefined terms 0, number, and successor in terms of simpler concepts although we may know what we mean by them. Russell says: "It is quite legitimate to say this (the last statement) when we must, and at *some* point we all must; but it is the object of mathematical philosophy to put off saying it as long as possible. By the logical theory of arithmetic we are able to put it off for a very long time."

In Chapters II and III the logical theory of the natural numbers is developed. Chapter II contains an exposition of the definition of cardinal number given by Frege, i.e., the cardinal number of a class is the class of all those classes that are similar to it. The following chapter is headed "Finitude and Mathematical Induction." There the definitions of 0, successor, hereditary property, hereditary class, inductive property, inductive class and the posterity of a natural number are given in terms of elemental logical concepts. The "natural numbers" are defined as the posterity of 0 with respect to the relation "immediate predecessor." The idea back of this procedure is that of mathematical induction. Russell emphasizes the fact that mathematical induction is a definition and not a principle. "There are some numbers to which it can be applied and there are others to which it cannot be applied. We *define* the 'natural numbers' as those to which proofs by mathematical induction can be applied, i.e., as those that possess all inductive properties." For this reason the author prefers the term "inductive numbers" to "natural numbers." Of course, this point of view is legitimate on the basis of the procedure above outlined. With respect to another setting up of the natural numbers mathematical induction might well be a principle.

Chapter IV is devoted to order relations and Chapters V and VI to relations in general. In the treatment of order the three kinds of relations, asymmetrical, transitive and connected, are defined preliminary to giving the following definition: A series or a serial relation is a relation which is asymmetrical, transitive and connected. On the basis of these definitions it is shown how the "natural numbers" can be ordered serially. Other examples of series are also given. Relations which do not have the three characteristic properties of serial relations are discussed and the chapter closes with a brief account of series for which the defining relation is between more than two terms. The relation "between" is discussed in some detail. Chapter V treats in a general way of relations. Neither here nor anywhere else in the book is "relation" defined. A clear cut definition of relation, say as a correspondence or as the underlying propositional function, and its discussion would seem to be essential to a treatment of relations such as given in this book. The author does not make this omission in the Principia (volume 1) and there seems to be no good reason for making it here. The necessary material is right at hand. The material of the present chapter is largely a repetition of matter which appeared in previous chapters, but its importance warrants repetition. Relations of the following kinds are considered: asymmetrical, transitive, connected, ancestral, one-one, one-many, many-one, many-many. In discussing the similarity of relations in Chapter VI the following two definitions are fundamental: "A relation S is said to be a correlator of two relations P and Q if S is one-one, has the field of Q for its converse domain and is such that P is the relative product of S and Q and the converse of S." "Two relations P and Q are similar if there exists at least one correlator of P and Q." When two relations are similar they share all properties which do not depend upon the actual terms in their fields. In this connection the question arises: "Given some statement in a language of which we know the grammar and the syntax, but not the vocabulary, what are the possible meanings of such a statement, and what are the meanings of the unknown words that would make it true?" This question is important because "it represents, much more nearly than might be supposed, the state of our knowledge of nature."

The notion of similarity leads to the concept "the relation

number of a given relation": the class of all those "relations that are similar to the given relation." The ordinal numbers are special cases of relational numbers. The chapter closes with an interesting application to the philosophical speculation concerning a comparison between an objective and a subjective world.

In Chapter VII the idea of number is extended by supplying logical definitions of rational, real and complex numbers. The author remarks that the discovery of correct definitions in this field was delayed by the common idea that each extension of number included the previous sorts as special cases. The definition of positive and negative integers, which is here given, is: "If m is any inductive number (natural number) then $+m$ is the relation of $n+m$ to n for any n (a cardinal number) and $-m$ is the converse relation, i.e., the relation of n to $n+m$." According to this definition "$+m$ is every bit as distinct from m as $-m$ is." A definition of a similar sort is given for positive and negative ratios. Definitions of the following terms are then given: "upper limit (lower limit) of a class α with respect to a relation P"; "maximum (minimum) of a class α with respect to a relation P"; "upper (lower) boundary of a set α." A "real number" is a segment of the series of ratios in order of magnitude. An "irrational number" is a segment of the series of ratios which has no boundary. A "rational number" is a segment of the series of ratios which has a boundary. In these definitions a segment is that class of the two determined by a Dedekind cut which contains the smaller numbers. A complex number is defined as an ordered pair of real numbers. The various arithmetical operations are defined and discussed for each particular class of numbers. The extensions in this chapter do not involve infinity.

In the next two chapters the notion of number is applied to infinite collections. On the basis of the assumption that no two inductive numbers have the same successor (given by Peano) it is shown that the number of inductive numbers is a new number not possessing all inductive properties. To quote the author again: "The difficulties that so long delayed the theory of infinite numbers were due largely to the fact that some, at least, of the inductive properties were wrongly judged to be such as *must* belong to all numbers; indeed it was thought that they could not be denied without contra-

diction. The first step in understanding infinite numbers consists in realizing the mistakenness of this view." The course of the discussion of this chapter leads naturally to the definitions: A reflexive class is one which is similar to a proper part of itself; a reflexive cardinal number is the cardinal number of a reflexive class. In order to give a definition of the number of inductive numbers the following definition of a progression is given: A progression is a one-one relation such that there is just one term belonging to the domain but not to the converse domain and the domain is identical with the posterity of this one term. The number of inductive numbers, $\aleph_0$, is the set of all domains of progressions. Some properties of $\aleph_0$ are developed. A finite class or cardinal is defined as one which is inductive and an infinite class or cardinal is one which is not inductive. The statement is made without proof that all reflexive classes are infinite (non-inductive). The reader is referred to a later chapter for the connection between the theorem that all infinite classes are reflexive and the multiplicative axiom. The higher transfinite cardinals and ordinals are briefly discussed and Chapter IX closes with a review of the formal laws obeyed by the transfinite cardinals and ordinals. A general definition of a transfinite ordinal is not given. "The importance of ordinals, though by no means small, is distinctly less than that of cardinals, and is very largely merged in that of the more general conception of relation-numbers."

Limits and continuity are the topics discussed in the next two chapters. The ordinal character of the notion of limit is emphasized. In the first of these chapters the notion of the limit of a set of elements and such related notions as minima, maxima, sequents, precedents, upper limits, lower limits and boundaries of a class with respect to a given relation are defined. A brief treatment of the Dedekind and Cantor definitions of continuous series is given at the end of the chapter. The other chapter is devoted to limits and continuity of functions and is more technical. The ordinary definitions of $\lim_{x \to a} f(x)$ and continuous function are given, though sometimes in a more abstract form.

The rest of the book is devoted to the logic (proper) of mathematics, the various topics treated becoming more and more elemental as the end of the book is reached. In Chapter

XII a very clear discussion of the multiplicative axiom is given. It is shown how this axiom or a weaker form is needed to prove such theorems as these: that any class can be well-ordered; that the sum of $\aleph_0$ classes of $\aleph_0$ members each has $\aleph_0$ members; that a non-inductive class is reflexive. The author's reaction to questions in this chapter is contained in the closing paragraph: "It is not improbable that there is much to be discovered in regard to the topics discussed in the present chapter. Cases may be found where propositions which seem to involve the multiplicative axiom can be proved without it. It is conceivable that the multiplicative axiom in its general form may be shown to be false. From this point of view, Zermelo's theorem offers the best hope: the continuum or some still more dense series *might* be proved to be incapable of having its terms well ordered, which would prove the multiplicative axiom false, in virtue of Zermelo's theorem. But so far, no method of obtaining such results has been discovered, and the subject remains wrapped in obscurity."

The subject of the next chapter is "The Axiom of Infinity and Logical Types." One form of the axiom of infinity is "If n be any inductive cardinal number, there is at least one class of individuals having n terms." Without the axiom of infinity or its equivalent the theory of real numbers and the theory of transfinite numbers would not exist. Russell spends some time showing that the axiom of infinity cannot be proved after postulating a class of individuals by forming the complete set of individuals, classes, classes of classes, etc. This kind of reasoning leads to such contradictions as the existence of the greatest cardinal number, the class of all classes, etc. The fallacy of the reasoning consists in the formation of a "class which is not pure as to type." At this point a little is said about the theory of types but the mention is too brief to be satisfying. A good brief exposition of the theory of types is probably impossible at this time. Some pertinent remarks are: "Classes are logical fictions and a statement which appears to be about a class will only be significant if it is capable of translation into a form in which no mention is made of the class." "If there are n individuals in the world and 2^n classes of individuals we cannot form a new class, consisting of both individuals and classes and having $n + 2^n$ members." The author does not pretend to have

explained the doctrine of types, but his object is to indicate why there is need for such a doctrine. Other "proofs," more or less metaphysical, of the axiom of infinity are briefly examined.

The next four chapters are the most fundamental of the book. Their object is a critique of the notion of *class*. The topics of the first three of these chapters, viz: (1) the theory of deductions and incompatibility, (2) propositional functions, (3) descriptions, although very important in themselves, are introductory to the study of the theory of classes given in the last of these chapters.

In the chapter on the theory of deduction Russell restates his thesis that "what can be known, in mathematics and by mathematical methods, is what can be deduced from pure logic." The essential part of the chapter consists of the definitions of the "truth-functions": "not-p" (negation); "p or q" (disjunction); "p and q" (conjunction); "p and q are not both true" (incompatibility); "not-p or q" (implication). All five truth-functions are not independent. Two, negation and disjunction, were chosen in the Principia Mathematica as fundamental and the others defined in terms of these. Sheffer has shown that one primitive idea is sufficient for that purpose. It is here shown that the single primitive idea of incompatibility is sufficient. An analysis of deduction is made on the basis of the five formal principles of deduction given in the Principia. A formal principle of deduction (e.g., "p or p implies p") has a double use: to serve as the premise of an inference or as a rule of deduction. A proof that the five formal principles can be reduced to one is given in detail. This single formal principle which is much more complicated, at least in statement, than any of the five is due to M. Nicod. This formal principle and two non-formal principles furnish the apparatus from which the whole theory of deduction follows "except in so far as we are concerned with deduction from or to the existence or the universal truth of propositional functions," which are studied in the next chapter. The chapter closes with an argument in support of the author's views on implication as against those of C. I. Lewis.

Of a propositional function it might be said that it is true in all cases or that it is true in some cases. The importance of this use of propositional functions is clearly pointed out

in Chapter XV. All the primitive propositions of logic as well as the principles of deduction consist of statements that certain propositional functions are always true. It is explained how the truth-functions as applied to propositions containing apparent variables can be defined in terms of the definitions and primitive ideas for propositions containing no apparent variables. For this it is found necessary to take as primitive ideas two of the following: "always," "sometimes," "not-ϕx sometimes" (or "always" as the case may be). The simpler forms of traditional formal logic really involve the assertion of all values or some values of a compound propositional function. For example: "all S is P" means "ϕx implies ψx is always true" where ϕx and ψx denote propositional functions. Russell's treatment of traditional logic leads to such results as the following: "if there are no S's then 'all S is P' and 'no S is P' will both be true, whatever P may be." Some reasons for preferring his treatment are given in convincing form.

The fundamental meaning of "existence" is contained in the following statement: If the propositional function ϕx is sometimes true we say that arguments satisfying ϕx *exist*. We may say men exist (here ϕx is x is a man) but it is nonsense to say John exists. Another instance of the use of propositional functions which we are considering is in the notions of "modality" (*necessary*, *possible* and *impossible*). An undetermined value of a propositional function ϕx is necessary if the function is always true, possible if sometimes true and impossible if it is never true. At the end of the chapter we have this sentence: "For clear thinking, in many diverse directions, the habit of keeping propositional functions sharply separated from propositions is of the utmost importance, and failure to do so in the past has been a disgrace to philosophy."

The theory of descriptions, treated in the next chapter, is very important from the point of view of logic and the theory of knowledge. Only those parts of the theory which are relevant to mathematics are here discussed. A proposition involving an indefinite description about "a so-and-so" is of the form "an object having the property ϕ has the property ψ" which means "The joint assertion of ϕx and ψx is not always false." It is an important point that such propositions contain no constituent represented by the phrase "a so-

and-so." Thus such propositions can be significant when there is no such thing as "a so-and-so." This is the solution of the philosophical question of "unreality" which Russell gives. The definition of propositions involving definite descriptions is: "the term satisfying ϕx satisfies ψx" which means that there is a term c such that (1) ϕx is always equivalent to 'x is c,' (2) ψc is true." The extra condition of uniqueness is added in this case.

The theory of classes, which is taken up in Chapter XVII, is concerned with the word *the* in the plural while that of definite descriptions deals with the singular meaning of that word. Because of the paradoxes involving the notion of class the latter cannot be taken as a primitive idea. It is desired to find "a definition which will assign a meaning to propositions in whose verbal or symbolic expression words or symbols apparently representing classes occur but which will assign a meaning that altogether eliminates all mention of classes from a right analysis of such propositions." The theory here outlined reduces propositions nominally about classes to propositions about the propositional functions which define them. The theory is incomplete because it is thrown back, in part, upon the incomplete theory of types. Because of this incompleteness it is found necessary to assume the axiom of reducibility: there is a type τ such that if ϕ is a function which can take a given object a as argument, then there is a function ψ of the type τ which is formally equivalent to ϕ. The fundamental definition of the theory of classes is: if ϕ is a function which can take a given object a as argument, and τ the type mentioned in the above axiom, then to say that the class determined by ϕ has the property f is to say that there is a function of type τ, formally equivalent to ϕ, and having the property f.

The final chapter "Mathematics and Logic" opens with an assertion of the Russellian thesis that logic and mathematics are identical. The proof in all detail is not given, but one is referred to the Principia. The remainder of the chapter is devoted to a discussion of what is characteristic of mathematical (or logical) propositions. Logical propositions affirm that some propositional function is always true. Specific propositions whose truth depends upon something else than the form of the propositions do not belong to mathematics but to its applications. Mathematical propositions have the characteristic described, perhaps, by the word "tautology."

This together with the fact that they can be expressed wholly in terms of variables and logical constants (invariants under all changes of the constituents of a proposition) gives a definition of logic or mathematics. The indefiniteness of the notion of tautology leaves this definition unsatisfactory and demands improvement. At this point, however, we reach the frontier of knowledge. It is pointed out that because of the insufficiency of language a symbolism for logic is necessary. At the close of the chapter the author expresses the hope that some who have read this book will master the symbolism of symbolic logic and then help to push back still further the frontier of knowledge, especially by a new treatment of the traditional problems of philosophy.

Throughout the exposition is clear and the style fluent. Here and there a refreshing bit of humor or sarcasm is thrown in. The book is written in a way which is as simple and untechnical as it is, perhaps, possible to be without sacrificing accuracy. Besides ably serving the purpose for which it was written, the book should be very useful to anyone who has a fundamental interest in science.

The following obvious errata in the text have been noted:

page 21, line 3 from bottom, "less than 1000" should read "not less than 1000";
page 33, line 7 from top, "but is asymmetrical" should read "but is not asymmetrical";
page 160, line 19 from top, "x is a mortal man" should read "x is not a mortal man";
page 161, line 2 from top, "to which x belongs" should read "to which ϕx belongs";
page 165, line 7 from top, "(where a is a term satisfying x)" should read "(where a is a term satisfying ϕx)";
page 171, line 5 from top, "the propositional function x" should read "the propositional function ϕx";
page 176, line 3 from bottom, "a propositional function x" should read "a propositional function ϕx";
page 176, line 1 from bottom, "the value of x" should read "the value of ϕx".

G. A. PFEIFFER.

NOTES.

The twenty-seventh annual meeting of the American Mathematical Society will be held in New York City on Tuesday and Wednesday, December 28–29. At this meeting President Morley will deliver his retiring address, the subject of which will be "Pleasant Questions and Wonderful Effects." The annual election of officers and other members of the Council will close on Wednesday morning. The regular western meeting of the Society, being the forty-sixth regular meeting of the Chicago Section, will be held at the University of Chicago, on December 29–30, in conjunction with the meetings of the Mathematical Association of America and the American Association for the Advancement of Science.

THE fifth summer meeting of the Mathematical Association of America was held at the University of Chicago on September 6, immediately preceding the meeting and colloquium of the American Mathematical Society. The attendance included 113 members. The following papers were read: "On certain fundamental principles in the mathematics of life insurance," by D. F. CAMPBELL; "Certain features of the application of Makeham's laws of mortality," by H. L. RIETZ; "The plan of pensions and insurance recommended by the Carnegie Foundation for the advancement of teaching," by E. L. DODD; Report of progress of the National committee on mathematical requirements, by J. W. YOUNG; "The debt of mathematics to the experimental sciences," by A. C. LUNN, followed by discussion, led by E. H. MOORE and M. W. HASKELL; "Retrospect and prospect for mathematics in America" (retiring presidential address), by H. E. SLAUGHT. Steps were taken towards the incorporation of the Association, which was completed under the laws of the state of Illinois in September. The Association held a joint dinner with the American Mathematical Society on Tuesday evening at the Quadrangle Club.

THE July number (volume 21, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "On the representation of a number as the sum of any number of squares, and in particular of five," by G. H. HARDY;

"A memoir upon formal invariancy with regard to binary modular transformations. Invariants of relativity," by O. E. GLENN; "Properties of the subgroups of an abelian prime power group which are conjugate under its group of isomorphisms," by G. A. MILLER; "On the order of magnitude of the coefficients in trigonometric interpolation," by DUNHAM JACKSON; "Concerning simple continuous curves," by R. L. MOORE; "On the iteration of rational functions," by J. F. RITT.

THE following fifteen doctorates with mathematics as major subject were conferred by American universities in the academic year 1919–1920; the title of the dissertation is added in each case: J. D. BOND, Michigan, "Plane trigonometry in Richard Wallingford's Quadri partium de sinibus demonstratis;" J. DOUGLAS, Columbia, "On certain two-point properties of general families of curves"; T. C. FRY, Wisconsin, "The use of divergent integrals in the solution of differential equations"; GLADYS GIBBENS, Chicago, "Comparison of different line-geometric representations for functions of a complex variable"; C. F. GREEN, Illinois, "On the summability and regions of summability of a general class of series of the form $\Sigma c_n g(x+n)$"; J. W. LASLEY, Chicago, "Some special cases of the flecnode transformation of ruled surfaces"; ELSIE J. MCFARLAND, California, "On a special quartic curve"; J. J. NASSAU, Syracuse, "Some theorems in alternates"; C. A. NELSON, Chicago, "Conjugate systems with conjugate axis curves"; E. L. POST, Columbia, "Introduction to a general theory of elementary propositions"; SUSAN M. RAMBO, Michigan, "The point at infinity as a regular point of certain difference equations of the second order"; L. L. STEIMLEY, Illinois, "On a general class of series of the form $Y(x) = C_0 + \Sigma C_n g(nx)$"; J. L. WALSH, Harvard, "On the location of the roots of the jacobian of two binary forms"; R. WOODS, Illinois, "The elliptic modular functions associated with the elliptic norm curve E^7"; T. YANG, Syracuse, "A problem in differential geometry."

The Library of the American Mathematical Society is in receipt of the thirteenth and final volume of the magnificent edition of the complete works of Christian Huygens issued by Nijhoff on behalf of the Dutch Society of Sciences. The

first ten volumes, containing the correspondence of Huygens during the period 1636–1695, were published from 1888 to 1905. The eleventh volume, appearing in 1908, was a comparatively small one of less than four hundred pages, and contained early mathematical works of Huygens of the period 1645–1651. The twelfth volume (1910), also a small one, was devoted to works on pure mathematics, 1652–1656. The thirteenth volume (1916), of over nine hundred pages, contains the various writings of Huygens on dioptrics during the period 1653–1692. It is doubtful if the works of any other scientist have received such detailed editing in the way of translations, footnotes, and all sorts of indexes.

In the department of mathematics of the University of Michigan the following promotions are announced: Professors W. H. Butts and T. R. Running to full professorships; Professors C. E. Love and T. H. Hildebrandt to associate professorships; Dr. L. J. Rouse, Dr. A. L. Nelson, Dr. W. W. Denton, and Dr. R. B. Robbins to assistant professorships. Dr. Robbins has been granted leave of absence for the year. Mr. N. Anning, Mr. S. E. Field, Mr. K. W. Halbert, Mr. G. D. Jones, Mr. J. N. Landis, and Mr. H. A. Simmons have been appointed instructors.

Professor J. M. Taylor, of Colgate University, has retired after fifty years of teaching.

Professor H. E. Buchanan, of the University of Tennessee, has been appointed head of the department of mathematics at Tulane University.

Professor E. W. Brown, of Yale University, has been granted leave of absence for the first half year. His address will be Christ's College, Cambridge.

Dr. A. S. Hathaway, professor of mathematics at the Rose Polytechnic Institute since 1891, has retired from active teaching. He is succeeded by Dr. I. P. Sousley, of Pennsylvania State College.

At the University of Iowa, assistant professor E. W. Chittenden has been promoted to an associate professorship

of mathematics; Dr. W. H. WILSON has been promoted to the rank of associate; Dr. ROSCOE WOODS, of the University of Illinois, has been appointed instructor in mathematics, and Mr. H. M. JEFFERS, of the Lick Observatory, instructor in mathematics and astronomy.

AT the University of Wisconsin, Professor C. S. SLICHTER has been appointed dean of the graduate school; associate professor E. B. SKINNER has been promoted to a full professorship; assistant professor H. W. MARCH has been promoted to an associate professorship; Dr. WARREN WEAVER has been appointed assistant professor; Mr. HAROLD DAVIS, of Harvard University, and Mr. M. L. MACQUEEN, of Southwestern Presbyterian University, have been appointed instructors in mathematics.

ASSISTANT professor EUGENE TAYLOR, of the University of Wisconsin, has been appointed professor and head of the department of mathematics at the University of Idaho.

PROFESSOR D. A. ROTHROCK has been elected dean of the college of liberal arts of Indiana University.

AT the University of Kentucky, Dr. F. ELIZABETH LE-STOURGEON, of Carleton College, has been appointed assistant professor of mathematics, and Mr. W. E. PAYNE has been appointed instructor. Dr. G. W. SMITH has resigned, to accept an assistant professorship of mathematics at the University of Kansas.

PROFESSOR PAUL BACHMANN died at Weimar, March 31, 1920, at the age of eighty-three years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BONOLA (R.). Die nichteuklidische Geometrie. Historisch-kritische Darstellung ihrer Entwicklung. Autorisierte deutsche Ausgabe, besorgt von H. Liebmann. 2te Auflage. Leipzig, Teubner, 1919. M. 6.40

DRAEGER (M.). Ueber rekurrente Reihen von höherer, insbesondere von der dritten Ordnung. (Dissertation.) Jena, 1919.

FISCHER (P.). Koordinatensysteme. 2te, verbesserte Auflage. Berlin, 1919. M. 1.25

FRICKE (R.). Hauptsätze der Differential- und Integralrechnung. 7te Auflage. Braunschweig, 1919. M. 7.60

INGHIRAMI (G.). Table des nombres premiers et de la décomposition des nombres de 1 à 100,000, revue et corrigée par le P. V. Prompt et suivie de la table des bases des nombres tessaréens de 1 à 20,000. Paris, Gauthier-Villars, 1919. 8vo. 20 + 36 pp. Fr. 7.50

LIEBMANN (H.). See BONOLA (R.).

MANN (W. D.). Das Preisproblem der Potenzenreihe $x^\lambda + y^\lambda - z^\lambda$ in Beziehung zu verwandten Problemen und Tatsachen der Gegenwart. Mexico, Deutsche Buchhandlung, 1919. $1.25

PAULY (J.). Notions élémentaires du calcul différentiel et du calcul intégral. Paris et Liège, Béranger, 1920. 8vo. 330 pp. Fr. 20.00

PRÖLSS (O.). Graphisches Rechnen. Leipzig, Teubner, 1920. M. 1.60

PROMPT (P. V.). See INGHIRAMI (G.).

SALMOIRAGHI (A.). Guida pratica del geometra moderno. 2a edizione. Milano, tip. U. Allegretti, 1920. 8vo. 24 + 120 pp. L. 10.00

ZSIGMONDY (K.). Zum Wesen des Zahlbegriffes und der Mathematik Antrittsrede, gehalten bei der feierlichen Rektors-Inauguration am 26. Oktober 1918 in der Technischen Hochschulen zu Wien. Wien, Verlag des Verfassers, 1919.

II. ELEMENTARY MATHEMATICS.

BARKER (E. H.). Applied mathematics for junior high schools and high schools. Boston, Allyn and Bacon, 1920. 12mo. 256 pp. $1.25

CAMINATI (C.) e CAMINATI (P.). Nuovo manuale italiano logaritmico-trigonometrico con sette o dieci decimali. Fascicolo 2: Tavole di logaritmi delle funzioni circolari a sette decimali, per uso delle scuole secondarie. Piacenza, tip. V. Porta, 1919. 8vo. 50 pp. L. 3.50

CAMINATI (P.). See CAMINATI (C.).

DURELL (C. V.) and PALMER (G. W.). Elementary algebra. Part 1. London, Bell, 1920. 8 + 256 + 8 pp. 3s. 6d.

LIETZMANN (W.). Riesen und Zwerge im Zahlenreich. 2te durchgesehene und vermehrte Auflage. (Mathematisch-physikalische Bibliothek, Nr. 25.) Leipzig, Teubner, 1919. 58 pp. M. 1.50

MILNE (R. M.). Mathematical papers for admission into the Royal Military Academy and the Royal Military College, and papers in elementary engineering for the Royal Air force for the years 1910–1919. Edited by R. M. Milne. London, Macmillan, 1920. 10s. 6d.

NEUENDORFF (R.). Lehrbuch der Mathematik. Für mittlere technische Fachschulen der Maschinenindustrie. 2te Auflage. Berlin, 1919. M. 12.00

PALMER (G. W.). See DURELL (C. V.).

WIELEITNER (H.). Die sieben Rechnungsarten mit allgemeinen Zahlen. 2te, durchgesehene Auflage. (Mathematisch-physikalische Bibliothek, Nr. 7.) Leipzig, Teubner, 1920. 55 pp.

III. APPLIED MATHEMATICS.

ARRHENIUS (S.). Der Lebenslauf der Planeten. Nach der 4ten Auflage aus den Schwedischen von B. Finkelstein. Leipzig, Akadem. Verl. Gesellschaft, 1919. Geb. M. 10.00

BAILLAUD (J.). Manuel de topométrie. Opérations sur le terrain et calculs. Paris, Dunod, 1920. 4to. 8 + 222 pp. Fr. 19.50

BERGET (A.). See CHAPPUIS (J.).

BIGOURDAN (G.). Petit atlas céleste précedé d'une introduction sur les constellations. 2e édition. Paris, Gauthier-Villars, 1920. 8vo. 56 pp. + 5 maps in colors. Fr. 3.00

BLONDEL (A.). See MAUDUIT (A.).

BUSCH (F.). Beobachtung des Himmels mit einfachen Instrumenten. 2te Auflage. (Mathematisch-physikalische Bibliothek, Nr. 14.) Leipzig, Teubner, 1919. 51 pp. Geh. M. 1.00

CAUNT (G. W.). See LOUIS (H.).

CHAPPUIS (J.) and BERGET (A.). Leçons de physique générale. Tome II: Electricité et magnétisme. 3é édition, entièrement refondue par J. Chappuis et M. Lamotte. Paris, Gauthier-Villars, 1920. 8vo. 624 pp. Fr. 45.00

DELLINGER (J. H.). Principles of radio transmission and reception with antenna and coil aerials. (Scientific Paper No. 354 of the Bureau of Standards.) Washington, Government Printing Office, 1919. 8vo. 64 pp. $0.10

DRZEWIECKI (S.). Théorie générale de l'hélice. Hélices aériennes et hélices marines. Paris, Gauthier-Villars, 1920. 8vo. 12 + 184 pp. Fr. 15.00

FABER (O.). See FERRARI (—.).

FERRARI (—.). Dioptric instruments; being an elementary exposition of Gauss's theory and its applications. Translated by O. Faber. London, Harrison, 1920. 31 + 214 pp. 4s.

FINKELSTEIN (B.). See ARRHENIUS (S.).

FÖPPL (A.). Vorlesungen über technische Mechanik. 3ter Band: Festigkeitslehre. 7te Auflage. 4ter Band: Dynamik. 5te Auflage. Leipzig, 1919. M. 16.00 + 16.00

HARTING (M.). See TURRIÈRE (E.).

HEIRMAN (E.). Calculs graphiques et analytiques du béton armé. Paris, Dunod, 1920. 4to. 208 pp. Fr. 18.00

HENKEL (O.). Graphische Statik mit besonderer Berücksichtigung der Einflusslinien. 2ter Teil. Berlin, 1919. M. 1.25

LAMOTTE (M.). See CHAPPUIS (J.).

LOUIS (H.) and CAUNT (G. W.). Tacheometer tables, with an introductory chapter on tacheometric surveying. London, Longmans, 1920. 8vo. 40 + 52 pp. 12s. 6d.

MASCART (J.). See REYNAUD (P.).

MAUDUIT (A.). Electrotechnique appliquée. Machines électriques (théorie, essais et construction). Cours professé à l'Institut électrotechnique de Nancy. Avec une préface de A. Blondel. Nouveau tirage. Paris, Dunod, 1920. 8vo. 20 + 930 pp. Fr. 52.50